JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **142**, 337–354 (1989)

Second-Order Right-Invariant Partial Differential Equations on a Lie Group

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Submitted by C. Foias

Received March 18, 1988

1. INTRODUCTION

The spectral theory of the right-invariant Laplace operator on a Lie group G was first considered in the papers [24, 22], by Nelson and Stinespring and by Nelson, respectively. The focus in these papers was the heat equation on G. Indeed, a fundamental solution convolution kernel, $p_t(g)$, $t>0$, $g \in G$, was obtained which generalizes the convolution kernel

$$p_t(x) = (4\pi t)^{-1/2}\, e^{-x^2/4t}, \qquad t>0,\, x \in \mathbb{R}$$

for the special case when $G = \mathbb{R}$. The case of G locally compact *abelian* goes back considerably further; see, for example, [4, 6] for references. In this case, the Fourier–Bessel–Hankel transform [5] provides a spectral resolution for the Laplace operator Δ on G. In case $G = \mathbb{R}^d$, the Radon transform provides a generalized eigenfunction expansion for the operator $\left(\begin{smallmatrix} 0 & 1 \\ \Delta & 0 \end{smallmatrix}\right)$ which is the infinitesimal generator of the solution group for the free wave equation [14, 17, 21].

Recently it has been shown that various curved magnetic field Hamiltonians can be solved explicitly using a sub-Laplacian on certain nilpotent Lie groups. In this paper, we give a simple, an relatively explicit, eigenfunction expansion for such sub-Laplacian operators. It turns out, for example, that more explicit results may be obtained for the sub-Laplacians than is the case for the corresponding elliptic Laplace operators defined from a basis for the Lie algebra. It follows, in particular, that the sub-Laplacians have no singular continuous spectrum when G is simply connected nilpotent. In fact, the spectrum is $[0, \infty)$, absolutely continuous, and with uniform multiplicity. More detailed information is contained in Theorem 4.1 below.

* Work supported in part by the NSF. The present paper was circulated as a preprint in the spring of 1985.

0022-247X/89 $3.00

2. Right-Invariant Partial Differential Operators

Let G be a Lie group with Lie algebra $\mathfrak{g}$, and exponential mapping exp: $\mathfrak{g} \to G$. Let U be a (strongly continuous) representation of G on a Hilbert space $\mathscr{H}$. It is well known that the infinitesimal representation dU is a representation of $\mathfrak{g}$ by skew-hermitian operators with the Gårding space as a dense invariant domain [11, 29, 24]. Indeed, if $\psi_1 = \int_G F(g)\, U(g)\psi\, dg$, for $F \in C_c^\infty(G)$, $\psi \in \mathscr{H}$, where dg is a left-invariant Haar measure on G, then

$$U(g)\psi_1 = \int_G (L(g)F)(g')\, U(g')\psi\, dg'$$

and

$$dU(X)\psi_1 = \int (dL(X)F)(g)\, U(g)\psi\, dg,$$

where

$$(L(g)F)(g') = F(g^{-1}g'), \qquad g, g' \in G, \tag{2.1}$$

$$dU(X)\psi_1 = \frac{d}{dt} U(\exp(tX))\psi_1 |_{t=0} \tag{2.2}$$

and

$$(dL(X)(g) = (\tilde{X}F)(g) = \frac{d}{dt} (L(\exp tX)F)(g)|_{t=0}$$

$$= \frac{d}{dt} F(\exp(-tX) \cdot g)|_{t=0}.$$

The collection of all vector fields $\tilde{X} = dL(X)$ for $X \in \mathfrak{g}$ exhausts the Lie algebra of all analytic right-invariant vector fields on G.

Similarly, dL extends to the complex universal enveloping algebra $\mathscr{U}(\mathfrak{g})$, and the collection of partial differential operators, $dL(T)$ for $T \in \mathscr{U}(\mathfrak{g})$, exhausts the algebra of all right-invariant partial differential operators on G.

The Nelson–Stinespring Laplace operator Δ arises this way from taking T to be the following second-order element in $\mathscr{U}(\mathfrak{g})$. Let $X_1, ..., X_d$ be a *basis* for $\mathfrak{g}$, and set $T = \sum_{i=1}^d X_i^2$. Then Δ is defined as

$$\Delta = dL(T) = \sum_{i=1}^d dL(X_i)^2 = \sum_{i=1}^d \tilde{X}_i^2. \tag{2.3}$$

It is well known to be elliptic although it is a variable coefficient operator.

If, for example, G is the $ax+b$ group with the parametrization $\begin{pmatrix} e^x & y \\ 0 & 1 \end{pmatrix}$, x, $y \in \mathbb{R}$, then we have the formula

$$\Delta = \left(\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2 \tag{2.4}$$

for the Laplace operator.

In this paper we shall be concerned with the case when $\mathfrak{g}$ is *nilpotent*. (Recall that the $ax+b$ Lie algebra is not. It is solvable.) Our *sub-*Laplacians will be constructed as follows: Start with a finite set of elements $X_1, ..., X_r$ in $\mathfrak{g}$ (generally *not* a basis), and impose the following two (relative mild) restricting assumptions:

Assumptions 2.1. (i) Assume that the elements X_i generate $\mathfrak{g}$ as a Lie algebra in the sense that $\mathfrak{g}$ is spanned by the X_i's *and* all iterated commutators, $[X_{i_1}, [X_{i_2}, [\cdots[X_{i_{n-1}}, X_{i_n}]\cdots]]]$, formed from the X_i's. Commutators of arbitrarily high order are admitted.

(ii) Assume there is a one-parameter family $\{\delta_s\colon s>0\}$ of Lie isomorphisms $\delta_s\colon \mathfrak{g} \to \mathfrak{g}$, such that $\delta_s(X_i) = sX_i$, $1 \leqslant i \leqslant r$.

We then say that the operator $\Delta = \sum_{i=1}^r \tilde{X}_i^2$ is a *sub-Laplacian.*

Remark 2.2. Assumption (i) is quite innocuous for the following reason. Suppose, for example, that $\{X_1, ..., X_r\}$ is an arbitrary finite subset of $\mathfrak{g}$. Then, let $\mathfrak{g}_1$ be the smallest Lie subalgebra of $\mathfrak{g}$ which contains this set, and let G_1 be the smallest closed subgroup of G containing $\{\exp(tX_i)\colon t \in \mathbb{R},\ 1 \leqslant i \leqslant r\}$. Then assumption (i) holds relative to $\mathfrak{g}_1$, and $\Delta = \sum_1^r \tilde{X}_i^2$ is hypoelliptic as a partial differential operator on G_1; cf. [18, 16, 12]. The problem of finding a fundamental solution for Δ has also been considered in a special case in [9].

3. The Spectrum

Let G be a Lie group with Lie algebra $\mathfrak{g}$. It was shown in [18] that every operator Δ on a Lie group G, of the form

$$\Delta = \sum_{i=1}^{r} \tilde{X}_i^2 \qquad \text{(where } \{X_i\} \text{ is an arbitrary subset of } \mathfrak{g}\text{)}, \tag{3.1}$$

is essentially self-adjoint on the space $C_c^\infty(G)$ when it is regarded as an operator in $\mathscr{L}^2(G)$ with domain of definition, $\mathscr{D}(\Delta)$, equal to $C_c^\infty(G)$. This means that the closure of Δ on $C_c^\infty(G)$ is self-adjoint and therefore given by a spectral resolution

$$-\Delta = \int_0^\infty \lambda \, dE(\lambda), \tag{3.2}$$

where $E(\cdot)$ is a projection valued measure supported on $[0, \infty)$. Hence, for $\psi \in \mathscr{L}^2(G)$, there is a monotone function h_ψ on $[0, \infty)$ of bounded variation such that

$$-(\Delta\psi, \psi) = \int_0^\infty \lambda \, \|dE(\lambda)\psi\|^2$$

$$= \int_0^\infty \lambda \, dh_\psi(\lambda),$$

where the integral relative to dh_ψ is the usual Riemann–Stieltjes integral and

$$\int_0^\infty dh_\psi(\lambda) = \|\psi\|^2.$$

In this paper, we find the spectral measure E, or, equivalently, the functions h_ψ in the special case where G is nilpotent.

The self-adjointness result from [18] applies in a much wider generality but E is not known in these more general cases. The spectral resolution of (2.3) in general, or even (2.4) in particular, is to our knowledge poorly understood.

It is well known (and immediate from the spectral theorem [7]) that the domain of Δ, now regarded as a (closed) self-adjoint operator with dense domain in $\mathscr{L}^2(G)$, consists precisely of functions $\psi \in \mathscr{L}^2(G)$ such that

$$\int_0^\infty \lambda^2 \, dh_\psi(\lambda) < \infty.$$

When the graph of Δ on $C_c^\infty(G)$ is closed up, the resulting operator, which we shall henceforth also denote by Δ, is self-adjoint. This is the content of the essential self-adjointness result [7, Theorem 1.1].

In the next section we consider generalized eigenfunctions $\varphi(\lambda, g)$ for Δ, i.e., functions φ on $[0, \infty) \times G$ such that

$$(\Delta\varphi)(\lambda, g) = -\lambda\varphi(\lambda, g), \qquad \lambda \geqslant 0, \, g \in G. \tag{3.3}$$

It follows from [15, 18] that every such function is C^∞ in the G-variable, and jointly measurable. We shall show that $\varphi(\lambda, \cdot)$ is never $\mathscr{L}^2$ on G. But there is a spectral representation R such that

$$(R\psi)(\lambda) = \int_G \psi(g) \, \varphi^*(\lambda, g) \, dg \tag{3.4}$$

is well defined. What this amounts to is that the functions $\varphi(\lambda, \cdot)$ may be chosen to have values in a fixed infinite-dimensional (when $\dim G > 2$) Hilbert space $\mathscr{V}$, and the integral in (3.4) is interpreted as a Bochner integral for functions (measurable as vector functions) with values in the Hilbert space $\mathscr{V}$. The norm on $\mathscr{L}^2(0, \infty; \mathscr{V})$ will be defined as

$$\|\varphi\|^2 = \int_0^\infty \|\varphi(\lambda)\|_{\mathscr{V}}^2 \frac{d\lambda}{\lambda}. \tag{3.5}$$

Recall that $d\lambda/\lambda$ is the Haar measure on the multiplicative group $\mathbb{R}_+$. An explicit formula for $\varphi(\lambda, \cdot)$, in the case of the Heisenberg group, is given in [31].

4. ANALYSIS OF THE SUB-LAPLACIAN

In this section we give our main result for the spectral transform which diagonalizes a sub-Laplacian $\Delta = \sum_{i=1}^r \tilde{X}_i^2$ on a simply connected nilpotent Lie group G. Recall that the right-invariant vector fields $\tilde{X}_i$ on G satisfy assumptions (i) and (ii) from Section 2. Only in the case when G is abelian is it the case that the X_i's form a basis for the Lie algebra $\mathfrak{g}$. But we have chosen that definition of the term "nilpotent" which excludes the abelian case. The result is still true of course for $G = \mathbb{R}^d$ (Euclidean space), and well known at that. We have chosen to briefly recall it so the reader can compare the nilpotent case to the flat one. For more details, see [21, 18, 5, 15]. At the same time, we take the opportunity to introduce terminology.

Let $G = \mathbb{R}^d$, and let $\Delta = \sum_{i=1}^d (\partial/\partial x_i)^2$ be the usual Laplace operator. Since Δ commutes with the $\mathcal{O}(d)$-action on $\mathbb{R}^d$, the spherical harmonics are components in the spectral decomposition of Δ on $\mathscr{L}^2(\mathbb{R}^d)$.

Let

$$\psi(x) = \sum_{n,j} Y_{nj}(\omega) f_n(r) \tag{$*$}$$

denote the spherical harmonics decomposition of $\psi \in \mathscr{L}^2(\mathbb{R}^d)$. Recall that $(*)$ is obtained by decomposing the unitary representation U given by

$$(U(A)\psi)(x) = \psi(A^{-1}x), \qquad A \in \mathcal{O}(d)$$

(of $\mathcal{O}(d)$ on $\mathscr{L}^2(\mathbb{R}^d)$) into irreducibles. We have used the usual convention for $x = r \cdot \omega$, where $r = (\sum_{i=1}^d x_i^2)^{1/2}$, and $\omega = r^{-1}x = (r^{-1}x_1, ..., r^{-1}x_d) \in S^{d-1}$. In view of $(*)$, it is enough to find the spectral decomposition relative to the radial part of the Laplace operator Δ.

It is classical [5] that the decomposition of a radial function, $f(x) = f(r)$, is given as

$$(Rf)(\lambda) = \int_0^\infty f(r)(2\pi r)^{d/2} \frac{J_{(d-2)/2}(r \cdot \lambda)}{\lambda^{(d-2)/2}}\, dr,$$

where $J_\mu(\cdot)$, $\mu = (d-2)/2$, denotes the Bessel function of the first kind. The densities,

$$\varphi(\lambda, r) = J_\mu(r \cdot \lambda)(r \cdot \lambda)^{-(d/2-1)},$$

are generalized eigenfunctions for Δ, i.e.,

$$\Delta\varphi(\lambda, \cdot) = -\lambda\varphi(\lambda, \cdot).$$

Introducing the decomposition $(*)$, we can define generalized eigenfunctions $\varphi(\lambda, x)$ with values in the Hilbert space $\mathscr{V} = \mathscr{L}^2(S^{d-1})$ where the inner product on $\mathscr{V}$ is defined relative to the normalized spherical measure $d\omega$ on S^{d-1}. Indeed, the function $\varphi(\lambda, \cdot)$, given as

$$x \to \varphi(\lambda, x) = (Y_{n,j}(\omega)\, J_\mu(r\lambda)(r\lambda)^{-d/2+1}),$$

may naturally be interpreted as defined on $\mathbb{R}^d$, $x = r \cdot \omega$, and taking values in the Hilbert space $\mathscr{V}$.

Our next theorem for the non-abelian sub-Laplacian is a close analogy to the classical Bessel transform.

THEOREM 4.1. *Let Δ be a sub-Laplacian on a simply connected nilpotent Lie group G. Then there is a separable Hilbert space $\mathscr{V}$, a conjugation, $v \to v^*$, and a unitary mapping R from $\mathscr{L}^2(G)$ to $\mathscr{L}^2(0, \infty; \mathscr{V})$ such that*

$$(R(\Delta\psi))(\lambda) = -\lambda(R\psi)(\lambda) \tag{4.1}$$

for al ψ in the domain of Δ, and

$$\int_G |\psi(g)|^2\, dg = \int \|(R\psi)(\lambda)\|_{\mathscr{V}}^2 \frac{d\lambda}{\lambda} \tag{4.2}$$

for all $\psi \in \mathscr{L}^2(G)$.

Moreover, there is a set of generalized eigenfunctions $\varphi(\lambda, g)$ with values in $\mathscr{V}$ such that

$$(R\psi)(\lambda) = \int_G \psi(g)\, \varphi^*(\lambda, g)\, dg, \tag{4.3}$$

where $\varphi^*(\lambda, g)$ is defined relative to a conjugation, $v \to v^*$, on $\mathscr{V}$.

Remark 4.2. A spectral representation R, with the properties (4.1) and (4.2) listed in the theore, may be constructed along the same lines, mutatis mutandis, for a more general class of right-invariant partial differential operators L on G. We list the three properties of L which allow us to construct such a spectral representation:

(1) $-\langle L\psi, \psi\rangle \geqslant 0$ for all $\psi \in C_c^\infty(G)$.

(2) We have, for some $d \in \mathbb{Z}_+$, $\delta_s(L) = s^d L$, $s \in \mathbb{R}_+$.

(3) For every irreducible representation π of G, different from the trivial one-dimensional representation, the operator $d\pi(L)$, defined on the C^∞-vectors for π, is invertible.

Remark 4.3. We work with generalized eigenfunction expansions in the usual sense of [11, 2, 8, 3, 23, 13]. Using the generalized eigenfunctions $\varphi(\lambda, g)$ from Theorem 4.1, we give the following direct integral decomposition of $\mathscr{L}^2(G)$: On the one hand, we have (cf. (3.2))

$$\psi = \int_0^\infty dE(\lambda)\psi, \qquad \psi \in \mathscr{L}^2(G),$$

and on the other hand, (4.1) and (4.2) translate into the decomposition

$$\psi(g) = \int_{\mathbb{R}_+} \int_G \psi(g')\langle \varphi(\lambda, g), \varphi(\lambda, g')\rangle_{\mathscr{V}}\, dg' \,\frac{d\lambda}{\lambda},$$

for $\psi \in \mathscr{L}^2(G)$, where $\langle\cdot,\cdot\rangle_{\mathscr{V}}$ denotes the inner product in the Hilbert space $\mathscr{V}$. But formula (4.3) allows us to write the decomposition in the more complete form

$$\psi(g) = \int_{\mathbb{R}_+} \langle (R\psi)(\lambda), \varphi^*(\lambda, g)\rangle_{\mathscr{V}} \,\frac{d\lambda}{\lambda},$$

which yields the explicit formula

$$dE(\lambda)\psi = \langle (R\psi)(\lambda), \varphi^*(\lambda, \cdot)\rangle_{\mathscr{V}} \,\frac{d\lambda}{\lambda}$$

for the spectral measure E of $-\Delta$.

Proof. The operator Δ is assumed to be a sub-Laplacian. By assumption (ii), Section 2, there is therefore a one-parameter family of Lie isomorphisms, $\delta_s\colon \mathfrak{g} \to \mathfrak{g}$, satisfying $\delta_s(X_i) = sX_i$, $1 \leqslant i \leqslant r$, $s \in \mathbb{R}_+$, where the elements $X_i \in \mathfrak{g}$ satisfy $\Delta = \sum_1^r \tilde{X}_i^2$. Since G is simply connected, $\{\delta_s\colon s \in \mathbb{R}_+\}$ exponentiates to a one-parameter group of Lie isomorphisms, $\sigma_s\colon G \to G$, satisfying $\sigma_s(\exp X) = \exp(\delta_s(X))$ for $X \in \mathfrak{g}$, and $s \in \mathbb{R}_+$. The Haar measure on G scales under σ_s according to the formula $d(\sigma_s(g)) = s^\nu\, dg$, where ν is an integer, $\nu = 0, 1, \ldots$, depending only on G and σ_s. Now define

$$(U_s\psi)(g) = \psi(\sigma_s(g))s^{\nu/2} \qquad \text{for} \quad \psi \in \mathscr{L}^2(G),\ g \in G,\ s \in \mathbb{R}_+. \tag{4.4}$$

It is immediate that U_s is a strongly continuous one-parameter family of unitary mappings, $U_s\colon \mathscr{L}^2(G) \to \mathscr{L}^2(G)$, and moreover U_s is a representation of the group $\mathbb{R}_+$ with multiplication.

From $\delta_s(X_i) = sX_i$, it is immediate that $\delta_s(\Delta) = s^2\Delta$. We claim that

$$U_s^*\,\Delta U_s = s^2\Delta, \qquad s > 0. \tag{4.5}$$

Indeed,

$$\begin{aligned}((U_s^*\,\Delta U_s)\psi)(g) &= (\Delta U_s\psi)(\sigma_s^{-1}(g))s^{-\nu/2}\\ &= s^2(\Delta\psi)(\sigma_s(\sigma_s^{-1}(g)))\\ &= s^2(\Delta\psi)(g) \qquad \text{for} \quad \psi \in \mathscr{L}^2(G),\ g \in G,\ s \in \mathbb{R}_+,\end{aligned}$$

which proves (4.5).

We noted in Section 3, (3.2), that the Spectral Theorem applies to the self-adjoint operator Δ, and that the spectral resolution may be expressed as

$$-\Delta = \int_0^\infty \lambda\, dE(\lambda) \tag{4.6}$$

with respect to a projection valued measure E with support in $[0, \infty)$. Substitution of (4.5) into (4.6) yields

$$\begin{aligned}\int_0^\infty \lambda\, dE(s^{-2}\cdot\lambda) &= \int_0^\infty s^2\cdot\lambda\, dE(\lambda)\\ &= -s^2\,\Delta = -U_s^*\,\Delta U_s\\ &= \int_0^\infty \lambda U_s^*\, dE(\lambda)\, U_s.\end{aligned}$$

By virtue of Lemma 4.4 below, we now conclude that

$$dE(s^{-2}\cdot\lambda) = U_s^*\, dE(\lambda)\, U_s, \tag{4.7}$$

i.e., the unitaries U_s scale the measure E.

LEMMA 4.4. *Let Δ be a sub-Laplacian as in the statement of Theorem 4.1. Then there is no nonzero $\psi \in \mathscr{L}^2(G)$ satisfying $(\Delta\psi, \psi) = 0$.*

Proof. Suppose some ψ satisfies the condition in the lemma. Then, by [15], we may conclude that $\psi \in C^{\infty}(G)$, and moreover [18] $\tilde{X}\psi \in \mathscr{L}^2(G)$ for all $X \in \mathfrak{g}$. It follows that

$$\begin{aligned}\sum_{i=1}^{r} \|\tilde{X}_i\psi\|^2 &= \sum_{1}^{r} (\tilde{X}_i\psi, \tilde{X}_i\psi) \\ &= -\sum_{1}^{r} (\tilde{X}_i^2\psi, \psi) \\ &= -(\Delta\psi, \psi) = 0,\end{aligned}$$

where $\|\cdot\|$, resp., $(\cdot,\cdot)$, denotes the norm, resp., the inner product of $\mathscr{L}^2(G)$. Hence, $\tilde{X}_i\psi = 0$ for $1 \leqslant i \leqslant r$, and therefore

$$[\tilde{X}_{i_1}, [\tilde{X}_{i_2}, \cdots [\tilde{X}_{i_{n-1}}, \tilde{X}_{i_n}] \cdots]]\psi = 0$$

for all indices i_j satisfying $1 \leqslant i_j \leqslant r$, and all $n = 2, 3, \ldots$. This implies that $\tilde{X}\psi = 0$ for all $X \in \mathfrak{g}$ by virtue of assumption (i), Section 2. Then ψ must be constant. But unless the constant is zero, this contradicts the assumed square-integrability of ψ. Recall that simply connected nilpotent Lie groups have infinite Haar measure [12].

Let $C_c(\mathbb{R}_+)$ be the space of continuous functions with compact support on the group $\mathbb{R}_+$, and define, for $\varphi \in C_c(\mathbb{R}_+)$, the mapping

$$P(\varphi) = \int \varphi(\lambda)\, dE(\lambda). \tag{4.8}$$

It follows from the Spectral Theorem that P is a representation of $C_c(\mathbb{R}_+)$, and in particular that P is positivity preserving, i.e., that $P(\varphi)$ is a positive operator on $\mathscr{L}^2(G)$ for all $\varphi \in C_c(\mathbb{R}_+)$, $\varphi \geqslant 0$. Indeed, if $\varphi \geqslant 0$, then

$$\begin{aligned}(P(\varphi)\psi, \psi) &= \int \varphi(\lambda)(dE(\lambda)\psi, \psi) \\ &= \int \varphi(\lambda)\, \|dE(\lambda)\psi\|^2 \geqslant 0\end{aligned}$$

for all $\psi \in \mathscr{L}^2(G)$.

Application of (4.7) in (4.8) yields the following covariance relation,

$$U_s^* P(\varphi) U_s = P(\tau_s\varphi) \tag{4.9}$$

for all $s \in \mathbb{R}_+$, $\varphi \in C_c(\mathbb{R}_+)$, where $\tau_s\varphi$ denotes translation on $\mathbb{R}_+$ by s^2, i.e., $(\tau_s\varphi)(\lambda) = \varphi(s^2 \cdot \lambda)$.

LEMMA 4.5. *The set* $P(C_c(\mathbb{R}_+))\mathscr{L}^2(G)$ *spans a dense subspace in* $\mathscr{L}^2(G)$.

Proof. Let $\zeta \in \mathscr{L}^2(G)$ be orthogonal to the set in the lemma relative to the inner product in $\mathscr{L}^2(G)$. The desired conclusion follows if we show that ζ must necessarily be the zero vector in $\mathscr{L}^2(G)$. By assumption, we have the identity $(P(\varphi)\psi, \zeta) = 0$ for all $\varphi \in C_c(\mathbb{R}_+)$, and $\psi \in \mathscr{L}^2(G)$.

We shall prove first that this implies that the measure $\lambda \to \|dE(\lambda)\zeta\|^2$ is supported in the single point, $\lambda = 0$.

Since E is supported on $[0, \infty)$, it follows that $\int_{\mathbb{R}} \varphi(\lambda)(dE(\lambda)\psi, \zeta) = 0$ whenever some $\varphi \in C_c(\mathbb{R})$ has support contained in $(-\infty, 0)$. Hence, it suffices to show that $\int \varphi(\lambda)\, \|dE(\lambda)\zeta\|^2 = 0$ whenever $\varphi \in C_c(\mathbb{R})$ has support contained in $(0, \infty)$. But such a φ may be regarded as an element in $C_c(\mathbb{R}_+)$, and we have $(P(\varphi)\psi, \zeta) = 0$. The density conclusion follows upon taking $\psi = \zeta$.

Hence, $\lambda \to \|dE(\lambda)\zeta\|^2$ is a point measure at $\lambda = 0$ [28], and the integral $\int \lambda\, \|dE(\lambda)\zeta\|^2$ is finite. Hence, $(\Delta\zeta, \zeta) = 0$. Lemma 4.4 now implies that $\zeta = 0$, and the proof of Lemma 4.5 is completed.

5. THE IMPRIMITIVITY THEOREM

In this section we point out that formula (4.9) together with Lemma 4.5 allows us to construct a spectral representation for Δ by applying the Mackey–Blattner imprimitivity theorem [1]. The form of that theorem, which is worked out in [25], and due to Poulsen and Ørsted, is particularly well suited for the present purpose.

We shall apply [25, Theorem 5] to the group $\mathbb{R}_+$ and the trivial subgroup $\Gamma = \{1\} \subset \mathbb{R}_+$. The assumption on P from [25, Theorem 5] is satisfied in our case by virtue of Lemma 4.5 above.

Let $\mathscr{D} \subset \mathscr{L}^2(G)$ be the Gårding space for the left-regular representation; cf. Section 2 (2.1). Then, for every pair of vectors $\psi_1, \psi_2 \in \mathscr{D}$, there is a continuous function h on $\mathbb{R}_+$ such that

$$(P(\varphi)\psi_1, \psi_2) = \int_{\mathbb{R}_+} \varphi(\lambda)\, h(\lambda)\, \frac{d\lambda}{\lambda} \tag{5.1}$$

for all $\varphi \in C_c(\mathbb{R}_+)$. Recall that $d\lambda/\lambda$ is the Haar measure on $\mathbb{R}_+$. The function $h(\lambda)$ depends, of course, on the pair of vectors $\psi_i \in \mathscr{L}^2(G)$, $i = 1, 2$.

The space $\mathscr{V}$ from Theorem 4.1 (Section 4) is defined as follows. First define a sequilinear (semi) inner product β on $\mathscr{D} \times \mathscr{D}$ by setting

$$\beta(\psi_1, \psi_2) = h_{\psi_1, \psi_2}(1), \tag{5.2}$$

where ψ_1, $\psi_2 \in \mathscr{D}$, and $h_{\psi_1,\psi_2}(\lambda)$ is the corresponding continuous function which is determined according to (5.1).

To get the Hilbert space $\mathscr{V}$, we mod out by the kernel of β, i.e., $\psi \in \mathscr{D}$, satisfying

$$\beta(\psi, \psi) = h_{\psi,\psi}(1) = 0;$$

and then, finally, we complete the quotient space, $\mathscr{V}/\ker\beta$, relative to the Hilbert norm defined by $\beta(\cdot,\cdot)$.

To get the conjugation, $v \to v^*$, on $\mathscr{V}$, we note that $\ker\beta$ is invariant under the usual conjugation on $\mathscr{D}$, i.e., $\psi \to \bar{\psi}$, where $\bar{\psi}(g) = \overline{\psi(g)}$. Indeed, we have $\beta(\psi, \psi) = \beta(\bar{\psi}, \bar{\psi})$ for all $\psi \in \mathscr{D}$, as can easily be checked from (5.1). It follows that $\psi \to \bar{\psi}$ passes to a conjugation on $\mathscr{V}$ which we shall denote $v \to v^*$. The essential step in the proof uses the simple fact that the original operator Δ commutes with $\psi \to \bar{\psi}$ on $\mathscr{L}^2(G)$.

Another application of (5.1) and (4.9) yields

$$h_{U_s\psi_1, U_s\psi_2}(\lambda) = h_{\psi_1,\psi_2}(s^{-2} \cdot \lambda). \tag{5.3}$$

For vectors $\psi \in \mathscr{D} \subset \mathscr{L}^2(G)$, we let $[\psi]$ denote the corresponding vector in $\mathscr{V}$. (Recall that a given ψ represents the zero vector in $\mathscr{V}$, i.e., $[\psi] = 0$, iff $h_{\psi,\psi}(1) = 0$.)

If we define

$$(R\psi)(\lambda) = [U_{\lambda^{-1/2}}(\psi)], \tag{5.4}$$

for $\psi \in \mathscr{D} \subset \mathscr{L}^2(G)$ and $\lambda \in \mathbb{R}_+$, then it follows from (5.3) above, and [25, Theorem 5], that R defines a mapping from $\mathscr{D}$ into $\mathscr{L}(\mathbb{R}_+, \mathscr{V})$ satisfying

$$\begin{aligned} &\int_{\mathbb{R}_+} \varphi(\lambda)\langle (R\psi_1)(\lambda), (R\psi_2)(\lambda)\rangle_{\mathscr{V}} \frac{d\lambda}{\lambda} \\ &\qquad = (P(\varphi)\psi_1, \psi_2) \qquad \text{for} \quad \varphi \in C_c(\mathbb{R}_+), \psi_1, \psi_2 \in \mathscr{D}, \end{aligned} \tag{5.5}$$

as well as

$$\int \|(R\psi)(\lambda)\|_{\mathscr{V}}^2 \frac{d\lambda}{\lambda} = \|\psi\|^2 \qquad \text{for} \quad \psi \in \mathscr{D}.$$

It follows further that R extends to a unitary mapping of $\mathscr{L}^2(G)$ onto $\mathscr{L}^2(\mathbb{R}_+, \mathscr{V})$ satisfying

$$(R(\Delta\psi))(\lambda) = -\lambda(R\psi)(\lambda) \qquad \text{for} \quad \psi \in \mathscr{D}. \tag{5.6}$$

Indeed, the latter formula (5.6) may be obtained from (5.1) above by approximation, taking into account Lemma 4.4.

This concludes the proof of formulas (4.1) and (4.2) in the statement of Theorem 4.1. We turn to (4.3) in the next section.

6. Generalized Eigenfunctions

We may now apply the *L*. Schwartz kernel theorem [30, 28, 13] to the operator[1]

$$R\colon \mathscr{L}^2(G) \to \mathscr{L}^2(\mathbb{R}_+, \mathscr{V}). \tag{6.1}$$

To see that the assumptions in Schwartz' theorem are satisfied we need to take advantage again of [18, Corollary 2.1]. It follows that there is a distribution kernel $K(\lambda, g)$, where K is a distribution on $\mathbb{R}_+ \times G$ with values in $\mathscr{V}$. Moreover, for $\psi \in C_c^\infty(G)$ and $\varphi \in C_c^\infty(\mathbb{R}_+, \mathscr{V})$ we have

$$\int \langle (R\psi)(\lambda), \varphi^*(\lambda) \rangle_{\mathscr{V}} \frac{d\lambda}{\lambda} = K(\psi \otimes \varphi). \tag{6.2}$$

We claim that $K(\lambda, \cdot)$ satisfies

$$\Delta K(\lambda, \cdot) = -\lambda K(\lambda, \cdot) \tag{6.3}$$

in the weak sense of distributions. Assuming (6.3) for the moment, then it follows that $K(\lambda, \cdot)$ is in $C^\infty(G)$ for each λ. Here [18, Corollary 2.1] is applied. The proof of (6.3) in turn follows from (6.2) and (5.6). Indeed,

$$\begin{aligned}
&\iint \langle (\Delta K)(\lambda, g)\, \psi(g), \varphi^*(\lambda) \rangle \, dg \frac{d\lambda}{\lambda} \\
&\quad = \iint \langle K(\lambda, g)(\Delta\psi)(g), \varphi^*(\lambda) \rangle \, dg \frac{d\lambda}{\lambda} \\
&\quad = \int \langle (R\Delta\psi)(\lambda), \varphi^*(\lambda) \rangle \frac{d\lambda}{\lambda} \\
&\quad = -\int \lambda \langle (R\psi)(\lambda), \varphi^*(\lambda) \rangle \frac{d\lambda}{\lambda} \\
&\quad = -\int \lambda \int \langle K(\lambda, g)\, \psi(g), \varphi^*(\lambda) \rangle \, dg \frac{d\lambda}{\lambda}
\end{aligned}$$

holds for all $\psi \in C_c^\infty(G)$, $\varphi \in C_c(\mathbb{R}_+, \mathscr{V})$.

[1] An explicit formula for this spectral transform has been worked out by the authors of [19] for the case when G is the Heisenberg group; see [31].

Recall that the Hilbert space $\mathscr{V}$ comes equipped with a conjugation, $v \to v^*$, which is determined by

$$R(\bar{\psi})(\lambda) = (R\psi)(\lambda)^*. \tag{6.4}$$

If we set $\varphi(\lambda, g) = K(\lambda, g)^*$, the desired formula (4.3) from Theorem 4.1 follows from identity (6.2) above.

7. Representations of Δ

We proved in [19] that physical Hamiltonians H with polynomial magnetic fields and potentials may be represented as $H = -dU(\Delta)$ for special choices of the nilpotent Lie group G, and some unitary representation U. Indeed, we showed [19, Sect. 8] that it suffices to consider *monomial* representations U of G. Recall [27] that a monomial representation U of G is a unitary representation of G which is induced from a one-dimensional representation of a subgroup of G. In fact, the following special case is general enough to apply to a wide class of Schrödinger operators H: Let N be an abelian subgroup of G, and let χ be a character on N, i.e., a one-dimensional, continuous, unitary representation of N; we write $\chi \in \hat{N}$. Let G/N be the quotient space, and let the canonical mapping from G to G/N be denoted $g \to \dot{g}$. The invariant measure on G/N will be denoted $d\dot{g}$. Then the representation U^χ may be realized as the space of scalar-valued functions ψ on G satisfying,

$$\psi(g \cdot n) = \chi(n)\, \psi(g), \qquad g \in G, n \in N. \tag{7.1}$$

Measurability is assumed, and we complete relative to the norm

$$\left(\int_{G/N} |\psi(\dot{g})|^2 \, d\dot{g} \right)^{1/2}.$$

Finally, U^χ is defined to be the restriction of the left-regular representation to this space. (Recall that, if $\psi \in \mathscr{H}(U^\chi) =$ the representation space for U^χ, then $|\psi(\cdot)|^2$ may be regarded as a function on the quotient G/N. We have $(U_g^\chi \psi)(g') = (L_g \psi)(g') = \psi(g^{-1} \cdot g')$, for $\psi \in \mathscr{H}(U^\chi)$.)

Corollary 7.1. *Let H denote the self-adjoint operator (Hamiltonian)*

$$H = -dU^\chi(\Delta) \tag{7.2}$$

for a sub-Laplacian Δ on a simply connected nilpotent Lie group G, N an abelian closed subgroup of G, and $\chi \in \hat{N}$.

Then, if H has purely continuous spectrum, it follows that the spectrum is in fact absolutely continuous, measurable multiplicity; in particular, no singular continuous spectrum.

Proof. Let $K(\lambda, g)$ be the kernel for the spectral representation R in (6.1). Recall that K is specified by formula (6.2). We now define a new kernel $K^\chi(\lambda, g)$ as

$$K^\chi(\lambda, g) = \int_N \chi(n)\, K(\lambda, g \cdot n)\, dn. \tag{7.3}$$

Then it is easy to check that $K^\chi(\lambda, g)$ satisfies the convariance condition

$$K^\chi(\lambda, g \cdot n) = \overline{\chi(n)}\, K^\chi(\lambda, g). \tag{7.4}$$

We claim that the kernel $K^\chi(\lambda, g)$ defines a unitary transformation R^χ of $\mathscr{H}(U^\chi)$ onto a subspace of $\mathscr{L}^2(\mathbb{R}_+, \mathscr{V})$ of the following form.

Let $\lambda \to e(\lambda)$ be a function on $\mathbb{R}_+$ which is measurable and takes values in the lattice of self-adjoint projections in the Hilbert space $\mathscr{V}$. Let $\tilde{e}$ be the corresponding projection in the Hilbert space $\mathscr{L}^2(\mathbb{R}_+, \mathscr{V})$ which is given by

$$(\tilde{e}\varphi)(\lambda) = e(\lambda)\, \varphi(\lambda) \tag{7.5}$$

for all $\varphi \in \mathscr{L}^2(\mathbb{R}_+, \mathscr{V})$. Then we show that R^χ maps $\mathscr{H}(U^\chi)$ onto a closed subspace of $\mathscr{L}^2(\mathbb{R}_+, \mathscr{V})$ of the form

$$\tilde{e}(\mathscr{L}^2(\mathbb{R}_+, \mathscr{V})), \tag{7.6}$$

where the decomposable projection $\tilde{e} = e(\lambda))$ depends only on χ.

The kernel $K^\chi(\lambda, g)$ from (7.3) determines an operator, which we shall denote by R^χ:

$$R^\chi \colon \mathscr{H}(U^\chi) \to \mathscr{L}^2(\mathbb{R}_+, \mathscr{V}).$$

Indeed, following (6.2), we use

$$(R^\chi \psi)(\lambda) = \int_{G/N} \psi(g)\, \varphi^*(\chi, \lambda, g)\, d\dot{g}, \tag{7.7}$$

where $\varphi(\chi, \lambda, g) = K^\chi(\lambda, g)^*$. Note that (7.6) is defined for $\psi \in \mathscr{H}(U^\chi)$ since the integrand,

$$g \to \psi(g)\, \varphi^*(\chi, \lambda, g),$$

then passes to the quotient G/N. Recall that, for $g \in G$, $n \in N$, we have $\psi(g \cdot n)\, \varphi^*(\chi, \lambda, g \cdot n) = \psi(g)\, \chi(n)\, \overline{\chi(n)}\, \varphi^*(\chi, \lambda, g)$ since

$$K^{\chi}(\lambda, g \cdot n) = \overline{\chi(n)}\, K^{\chi}(\lambda, g)$$

by (7.3).

To begin with, we define (7.6) just for $\psi \in C^{\infty}(G) \cap \mathscr{H}(U^{\chi})$ such that $|\psi(\cdot)|^2$ is of compact support on G/N. Formula (7.6) will then be understood in the sense of distributions.

On the other hand, we have, for fixed $\lambda \in \mathbb{R}_+$,

$$\begin{aligned} HK^{\chi}(\lambda, g) &= -dU^{\chi}(\Delta)\, K^{\chi}(\lambda, g) \\ &= -\int_N \chi(n)\, \Delta K(\lambda, g \cdot n)\, dn \\ &= \lambda \int_N \chi(n)\, K(\lambda, g \cdot n)\, dn \\ &= \lambda K^{\chi}(\lambda, g), \end{aligned}$$

where we have used that $\Delta = \sum_{i=1}^{r} \tilde{X}_i^2$ is a right-invariant partial differential operator on G. It follows that $K^{\chi}(\lambda, \cdot)$ is a generalized eigenfunction for H, and therefore smooth in the second variable.

Using Fubini's theorem, we can now show that

$$\int_{\mathbb{R}_+} \|(R^{\chi}\psi)(\lambda)\|_{\mathscr{V}}^2 \frac{d\lambda}{\lambda} = \|\psi\|_{\mathscr{H}(U^{\chi})}^2, \tag{7.8}$$

where $(R^{\chi}\psi)(\lambda)$ is given by (7.6).

It follcws that R^{χ} is a spectral representation for

$$H = -dU^{\chi}(\Delta).$$

Since $\mathscr{H}(U^{\chi})$ is invariant under the action of Δ, and therefore of $P(\varphi)$ from (4.8), we conclude that the range of R^{χ} is invariant under multiplication by scalar functions, acting as operators on $\mathscr{L}^2(\mathbb{R}_+, \mathscr{V})$, i.e., the range is decomposable. Hence, it is of the form (7.6) where $\tilde{e}$ is specified in (7.5), by virtue of a well-known characterization of decomposable projections; see, for example, [21]. (It is not known exactly how many decomposable projections $\tilde{e}$ do arise this way.)

8. AN EXAMPLE

In this final section, we recall some details from an example in [19]. The example illuminates both our present Theorem 4.1 and Corollary 7.1. In

[19], the three-dimensional magnetic field B given by $B_x = a_0 x$, $B_y = -a_0 y$, $B_z = 0$ is considered, and the vector potential $A_x = A_y = 0$, $A_z = a_0 xy$ is chosen. We solve *explicitly* the Schrödinger equation in $\mathscr{L}^2(\mathbb{R}^3)$ for the Hamiltonian H given by

$$-H = \left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2 + \left(\frac{\partial}{\partial z} - i\gamma xy\right)^2,$$

where $\gamma \in \mathbb{R}$ is dimensionless and gives the field strength.

Associated to H we have the nilpotent group G of upper triangular real 4×4 matrices, $g = g(a, b, c)$, where

$$g = \begin{pmatrix} 1 & a_1 & b_2 & c \\ & 1 & a_3 & b_1 \\ & & 1 & a_2 \\ \bigcirc & & & 1 \end{pmatrix}.$$

For the sub-Laplacian Δ we have

$$\Delta = \sum_{i=1}^{3} \tilde{X}_i^2,$$

where

$$-\tilde{X}_1 = \frac{\partial}{\partial a_1} + a_3 \frac{\partial}{\partial b_2} + b_1 \frac{\partial}{\partial c},$$

$$-\tilde{X}_2 = \frac{\partial}{\partial a_2},$$

and

$$-\tilde{X}_3 = \frac{\partial}{\partial a_3} + a_2 \frac{\partial}{\partial b_1}.$$

In this example, the induced representation U^χ (cf. Section 7) is induced from $N_3 = \{g(a, b, c) : a = 0\} \cong \mathbb{R}^3$ via $\chi \in \hat{N}_3$, $\chi = (0, 0, -\gamma)$, and U^χ may be realized explicitly as a multiplier representation acting on $\mathscr{L}^2(\mathbb{R}^3)$, where $\mathbb{R}^3$ is parametrized by the physical variables (x, y, z). $-H = dU^\chi(\Delta)$ has absolutely continuous spectrum. We show [19] that U^χ in turn decomposes as a direct integral (via Fourier transform in one variable). The irreducible components are again induced, but now form

$$N_4 = \{g(a, b, c) : a_1 = a_2 = 0\} \cong \mathbb{R}^4,$$

i.e., N_4 is parametrized by (a_3, b_1, b_2, c). The elements $\xi \in \hat{N}_4$ which contribute (relative to the Plancherel measure) are given by the parametrization $\xi = (\alpha_3, 0, 0, -\gamma) \in \hat{N}_4 \cong \mathbb{R}^4$, and moreover,

$$dU^{\xi}(\Delta) = \left(\frac{\partial}{\partial x_1}\right)^2 + \left(\frac{\partial}{\partial x_2}\right)^2 - (\alpha_3 + \gamma x_1 x_2)^2.$$

This means that $H_{\xi} = -dU^{\xi}(\Delta)$ is again a Schrödinger operator, for each ξ, but now it acts only in the two variables (x_1, x_2), as a self-adjoint operator in $\mathscr{L}^2(\mathbb{R}^2)$ with purely discrete spectrum. The detailed results for $H_{\xi} = -dU^{\xi}(\Delta)$ are quite explicit, and the reader is referred to [19] for precise formulas.

Remark 8.1. Even though the sub-Laplacian Δ always has Lebesgue spectrum as an operator on $\mathscr{L}^2(G)$ when G is nilpotent, it may happen that for some representation U^{χ} (as in Section 7), the operator $-dU^{\chi}(\Delta)$ has discrete spectrum. This happens also for the constant magnetic field [19]. In this case, we may take G to be the Heisenberg group. The reader is again referred to [19] for details.

Finally, we refer to [20, Chaps. 4 and 10] for two different applications of sub-Laplacians.

In [26], Penny considers geometric applications of a similar class of right-invariant operators Δ_{ε} on G. He shows that, if $\Delta_{\varepsilon} = \sum \varepsilon_i \tilde{X}^2$ with $\varepsilon_i = \pm 1$ well behaved relative to a complex structure, then Δ_{ε} has a particularly simple spectral theory.

References

1. R. J. Blattner, *in* "Harmonic Analysis and Representations of Semisimple Lie Groups" (J. A. Wolf *et al.*, Eds.), pp. 1–67, Reidel, Dordrecht/Boston, 1980.
2. F. E. Browder, The eigenfunction expansion theorem for the general self-adjoint singular elliptic partial differential operator, I, II, *Proc. Math. Acad. Sci. USA* **40** (1954), 454–467; **42** (1956), 769–771, 870–872.
3. W. G. Bade and J. T. Schwartz, On Mautner's eigenfunction expansion, *Proc. Nat. Acad. Sci. U.S.A.* **42** (1956), 519–525.
4. C. Berg and G. Forst, "Potential Theory on Locally Compact Abelian Groups," Springer-Verlag, Berlin/New York, 1975.
5. S. Bochner and K. Chandrasekharen, Fourier Transforms," Annals of Mathematics Studies, Vol. 19, Princeton Univ. Press, Princeton, NJ, 1949.
6. O. Bratteli and P. E. T. Jorgensen, *in* "Positive Semigroups of Operators and Applications," *Acta Appl. Math.* **2**, Nos. 3–4 (1984), 213–219 (Special Issue).
7. N. Dunford and J. T. Schwartz, "Linear Operators," Vol. II, Interscience, New York, 1963.
8. C. Foiaş, Problèmes mixtes et décompositions en noyaux propres des opérateurs aux dérivées partielles, *C. R. Acad. Sci. Paris* **255** (1962), 247–248.

9. G. B. FOLLAND, Subelliptic estimates and function spaces on nilpotent Lie groups, *Ark. Mat.* **13** (1975), 161–208.
10. L. GÅRDING, Note on continuous representations of Lie groups, *Proc. Nat. Acad. Sci. U.S.A.* **33** (1947), 331–332.
11. L. GÅRDING, *in* "Partial Differential Equations" (L. Bers, F. John, and M. Schechter, Eds.), pp. 303–325, Amer. Math. Soc., Providence, RI, 1964.
12. R. W. GOODMAN, "Nilpotent Lie Groups," Lecture Notes in Mathematics, Vol. 562, Springer-Verlag, New York, 1976.
13. I. M. GEL'FAND, M. I. GRAEV, AND N. YA. VILENKIN, "Generalized Functions," Vol. 5, Academic Press, New York, 1966.
14. S. HELGASON, "The Radon Transform," Birkhäuser, Boston, 1980.
15. L. HÖRMANDER, Hypoelliptic second order differential equations, *Acta Math.* **119** (1968), 147–171.
16. H. HULANICKI, The distribution of energy in the Brownian motion inthe Gaussian field and analytic hypoellipticity of certain subelliptic operators on the Heisenberg group, *Studia Math.* **56** (1976), 165–173.
17. F. JOHN, "Plane Waves and Spherical Means Applied to P.D.E.," Wiley–Interscience, New York, 1955.
18. P. E. T. JORGENSEN, Representations of differential operators on a Lie group, *J. Funct. Anal.* **20** (1975), 105–135.
19. P. E. T. JORGENSEN AND W. H. KLINK, Nilpotent Lie groups and Schrödinger operators. I. The curved magnetic field, preprint, 1985; *Publ. Res. Inst. Math. Sci.* **21** (1985), 969–999.
20. P. E. T. JORGENSEN AND R. T. MOORE, "Operator Commutation Relations," Reidel, Dordrecht/Boston, 1984.
21. P. D. LAX AND R. S. PHILLIPS, "Scattering Theory," Academic Press, New York, 1967.
22. E. NELSON, Analytic vectors, *Ann. of Math.* **70** (1959), 572–615.
23. E. NELSON, Kernel functions and eigenfunction expansions, *Duke Math. J.* **25** (1958), 15–28.
24. E. NELSON AND W. STINESPRING, Representation of elliptic operators in an enveloping algebra, *Amer. J. Math.* **81** (1959), 545–560.
25. B. ØRSTED, Induced representations and a new proof of the imprimitivity theorem, *J. Funct. Anal.* **31** (1979), 355–359.
26. R. C. PENNEY, Non-elliptic Laplace equations on nilpotent Lie groups, *Ann. of Math.* **119** (1984), 309–385.
27. L. PUKANSKY, "Leçons sur les représentations des groupes," Dunod, Paris, 1967.
28. L. SCHWARTZ, "Théorie des distributions, I, II," Hermann, Paris, 1957, 1959.
29. I. E. SEGAL, A theorem on the measurability of group invariant operators, *Duke Math. J.* **26** (1959), 549–552.
30. R. F. STREATER AND A. S. WIGHTMAN, "P.C.T. Spin and Statistics and All That," Benjamin, New York, 1964.
31. P. E. T. JORGENSEN AND W. H. KLINK, Spectral transform for the sub-Laplacian on the Heisenberg group, *J. d'Analyse Math.* **51** (1988).